Indian Statistical Institute, Bangalore
M. Math. Second Year
First Semester - Advanced Probability

Backpaper Exam — Date : Jan 07, 2014
Max Marks:45 — Time:3 hours

1. Let X be a r.v on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G} \subset \mathcal{F}$ be as sub σ-field. Let $X \in L^1(P)$.
   (a) If $X \geq 0$, show that $E[X \mid \mathcal{G}] \geq 0$ almost surely. [5]
   (b) If Y is a bounded $\mathcal{G}$ measurable random variable, then show that $E(YX \mid \mathcal{G}) = YE(X \mid \mathcal{G})$ a.s. [5]
   (c) Let $X = (X_1, ..., X_n) \in \{0,1\}^n$ and $Y \in [0,1]$ have the joint distribution
   $$P\{X = x, Y \in B\} = \int_B y^k (1-y)^{n-k} dy$$
   where $x = (x_1, ..., x_n) \in \{0,1\}^n$ and $k = \#\{i : x_i = 1\}$. Construct explicitly a regular conditional distribution of X given $Y = y \in [0,1]$. [5]

2. (a) Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter $\lambda > 0$. Show that $\{X_t - \lambda t, t \geq 0\}$ is a martingale with respect to the natural filtration of (X_t). [5]
   (b) Let $\{X_t, \mathcal{F}_t, t \geq 0\}$ be a sub-martingale and $\varphi : \mathbb{R} \to \mathbb{R}$ be non decreasing and convex. Show that $\{\varphi(X_t), \mathcal{F}_t\}$ is a sub-martingale. [5]

3. (a) Prove Slutsky's result: Let $(\mathcal{S}, d)$ be a metric space. Let X_n, Y_n and X take values in S. Suppose $X_n \to X$ in distribution and $d(X_n, Y_n) \to 0$ in probability. Then $Y_n \to X$ in distribution. [5]
   (b) Define probability measures $\mu_n, n \geq 1$ on $[0,1]$ as follows:
   $$\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{i}{n}}$$
   Then show that $\{\mu_n\}$ converges weakly to the Lebesgue measure on $[0,1]$. [5]

4. (a) Let X be an exponential random variable with parameter $\theta > 0$. Show that X is characterized by its moments i.e., if $EX^n = EY^n$ for all $n \geq 1$, where $Y \geq 0$ then $X \stackrel{d}{=} Y$. [5]
   (b) Let X have a finite second moment and let $\varphi(t)$ be the characteristic function of X. Show that $\varphi(t)$ is twice differentiable at the origin and $-\varphi''(0) = EX^2$ [5]

5. (a) Let $\{X_n, n \geq 1\}$ be a sequence of integrable random variables. Show that if the strong law holds for (X_n) then the weak law also holds. [5]
   (b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Let $\{X_n, n \geq 1\}$ be i.i.d, $X_1 \sim$ Ber(p) and let $S_n = X_1 + ... + X_n$. Show that $Ef(\frac{S_n}{n})$ is a polynomial in $p, 0 \leq p \leq 1$. [5]
   (c) Use (b) to prove Weierstrass's result viz., there exists a sequence of polynomials converging to f uniformly in $[0,1]$. [5]

6. Let $0 < p < 1$ and let μ_p be the probability measure on $\{0,1\}$ with $\mu_p\{1\} = p = 1 - \mu_p\{0\}$. Use the Kolmogorov consistency theorem to construct the infinite product measure $\bigotimes_1^\infty \mu_p$ on $\{0,1\}^N$. [10]